



CHARACTERIZATION OF VOLUME SCATTERING WAVENUMBER POWER
SPECTRA IN ISOTROPIC MEDIA FROM WAVENUMBER
POWER SPECTRA OF SCATTERING BY PLANES

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Report DURL 81-2

Diagnostic Ultrasound Research Laboratory
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by

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ABSTRACT

Fourier transform relations that exist between the correlation function of medium variations and the wavenumber spectrum of scattering are simplified for scattering by a volume and by a plane when the medium is statistically isotropic. The simplifications produce one-dimensional transform relations which are used to determine the wavenumber power spectrum of scattering by a volume from the power spectrum of scattering by a plane within the volume. Additionally, the isotropy is employed to compute moments that describe the wavenumber distribution of spectral power scattered by a volume in terms of moments describing the power scattered by a plane. As a part of the analysis, general relations between wavenumber power spectra of scattering in n , $n-1$, and $n-2$ dimensions are given, and corresponding relations for moments of scattered power are developed. Calculations of wavenumber power spectra and various moments for volume scattering are made from two-dimensional power spectra having analytic forms as well as two-dimensional wavenumber power spectra computed from planar sections of closely packed spheres and stained sections of pig liver tissue. Numerical results obtained by processing the two-dimensional wavenumber power spectra which yielded closed form expressions for volume scattering were in close agreement with theoretical values. Results of calculations for the packed spheres and the serial sections of liver agree reasonably well with predicted relationships and indicate how quantitative information about the three-dimensional structure can be obtained from two-dimensional sections when the basic isotropy conditions are satisfied.



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INTRODUCTION

The ultrasonic characterization of tissue from its scattering properties requires correlation of volume scattering data with structure that is presently observed optically in thin planar sections stained to enhance tissue components. The most common practice today is to correlate ultrasonically determined tissue parameters with optically determined tissue structure by using verbal descriptions of observations. This qualitative approach has merit because its use of language permits a great degree of flexibility to cover a wide variety of circumstances. However, a quantitative system is also desirable for correlation of acoustic volume scattering data with images of thin planar sections produced by light microscopy or the emerging field of acoustic microscopy.

This report describes analyses and calculations we have carried out to obtain a quantitative correlation. Our approach uses the wavenumber power spectrum of scattering to describe the structure of tissue. The approach applies equally well to either a totally acoustic or optical determination of structure. The approach also applies to correlation of acoustic wavenumber power spectra with optical wavenumber power spectra when the optical wavenumber range is comparable and there is a direct correspondence between structures that scatter acoustic waves and those that scatter light. When a subset of structures that scatter light can be selectively enhanced by special staining techniques or there are differences in optical and acoustic appearance, the approach may be employed to identify contributions to scattering by various tissue

components which can be visualized in one modality but not the other.

The objective of this paper is to develop and illustrate relations that exist between the wavenumber power spectrum of scattering by a plane and by a volume in a statistically isotropic medium. The analysis starts with a model that expresses an appropriately normalized mean square value or power spectrum of a scattered wave as the Fourier transform of the wave speed variations and uses the fact that the Fourier transform can be reduced to a one-dimensional expression when the region is statistically isotropic. Fourier projection relations are employed to write the power spectrum in spaces of $n-1$ and $n-2$ dimensions in terms of the power spectrum in an n -dimensional space from which they are derived. The various power spectrum expressions are used to determine a volume scattering power spectrum from the power spectrum of scattering by a plane. Moments that describe the distribution of scattered power are defined and, using projection formulas for power spectra, relations between moments in n , $n-1$ and $n-2$ dimensions are found. The power spectra and moment relations are illustrated and discussed for data from analytic functions and experimental models.

THEORY

The model we employ for scattering is analogous to that used in the description of x-ray diffraction by amorphous materials (1), the analysis of light diffraction by two-dimensional screens (2), and the characterization of atmospheric turbulence (3). In the model, the scattering is assumed to be caused by spatial variations in wave speed in a lossless unbounded medium. If a time harmonic plane incident wave is assumed, if the scattering is weak (i.e. the Born approximation is valid), and if the measurements are made in the far-field of the scattering volume, an expression for the scattered wave can be written as a frequency-dependent factor times the three-dimensional Fourier transform of the spatial variations in wave speed. Under the additional assumptions that the correlation function of the wave speed variations is independent of spatial origin, i.e. the medium is statistically homogeneous, and the distance over which scattering is correlated is small compared to the dimensions of the scattering volume, the mean square value of the scattered wave amplitude can be expressed as another frequency-dependent factor times the three-dimensional Fourier transform of the correlation function of the wave speed variations.* The result is (4)

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4 V}{4\pi^2 d^2} \int_V e^{j\underline{k} \cdot \underline{r}} B_{\chi_c}(\underline{r}) d^3 r \quad (1)$$

* see Appendix A.

where

ϕ_s = complex amplitude of scattered wave

$\underline{K}$ = scattering vector having three components

k = wavenumber

V = scattering volume

d = distance from scattering volume to observer

$$B_{\gamma_c}(\underline{r}) = \langle \gamma_c(\underline{r}') \gamma_c(\underline{r}' + \underline{r}) \rangle$$

γ_c = variation in wave speed

$\underline{r}, \underline{r}'$ = position vectors in the scattering volume

and $\langle \cdot \rangle$ denotes a statistical average. In these expressions, B_{γ_c} is a three-dimensional correlation function of the wave speed variations in the scattering volume which contains the three-dimensional vectors $\underline{r}$, and $\underline{r}'$. Since the average scattered wave intensity is proportional to the mean square value of the scattered wave amplitude and the Fourier transform of a process correlation function yields a power spectrum of that process, this Fourier transform relation shows that the intensity of the scattered wave is proportional to the wavenumber power spectrum of the variations in sound speed or, more briefly, the power spectrum of scattering.

Wave scattering by a plane can be described in terms of two-dimensional Fourier transform relations that are analogous to the

three-dimensional Fourier transform relations. The analogous two-dimensional expression for the mean square value of a wave scattered by a plane is

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4 A}{4\pi^2 d^2} \int_A e^{j\underline{k} \cdot \underline{r}} B_{\gamma_c}(\underline{r}) d^2 r \quad (2)$$

where

$\underline{k}$ = scattering vector having two components

A = planar area producing the scattering

d = distance from scattering plane to observer

$$B_{\gamma_c}(\underline{r}) = \langle \gamma_c(\underline{r}') \gamma_c(\underline{r}' + \underline{r}) \rangle$$

$\underline{r}, \underline{r}'$ = position vectors in the scattering plane

and

γ_c = wave speed variations within plane.

In these expressions, the quantity B_{γ_c} is the two-dimensional correlation function of the wave speed variations in the scattering plane which contains the two-dimensional vectors $\underline{r}_1$ and $\underline{r}'$.

This last Fourier transform expression can be obtained from a three-dimensional result for scattered wave amplitude by assuming the variation function to be non-zero only within a plane, representing the plane as a product of an impulse function and a two-dimensional variation function, and then integrating over the

coordinates of the impulse function. Alternatively, the two-dimensional Fourier transform expressions can be obtained by considering an infinite screen which transmits an incident wave amplitude proportional to the variation in the screen. This latter approach is often employed in scalar analysis of optical diffraction. In either development, assumptions of plane incident wave, weak single scattering, and observation far from the scattering plane are still required to obtain the two-dimensional expressions given here for scattered wave amplitude and average intensity. Additional assumptions that the wave speed variations are statistically stationary in space and have a correlation length which is short compared to the dimensions of the scattering region are also still required to get the two-dimensional result for average scattered intensity just as in the volume scattering case.

The mean square amplitude of the wave scattered by a volume and by a planar region can be normalized to eliminate the frequency-dependent factor and yield an expression which is exactly equal to the Fourier transform of the correlation function of the wave speed variations. For volume scattering, the normalization is

$$\Phi_v(\underline{k}) = \frac{\langle |\phi_s(\underline{k})|^2 \rangle}{2\pi k^4 V} d^2 \quad (3)$$

while, for scattering by a planar region, the normalization is

$$\Phi_P(\underline{k}) = \frac{\langle |\phi_s(\underline{k})|^2 \rangle}{k^4 A} d^2 \quad (4)$$

In each case, the resulting quantity is the differential scattering cross-section (i.e. the average power scattered into an aperture per unit values of incident intensity, scattering region and solid angle of aperture) divided by wavenumber raised to the fourth power.

Appropriately normalized versions of the mean square values of the wave amplitude scattered by regions of two or three dimensions are special cases of a Fourier transform which is defined in an n -dimensional real Cartesian space. The basic Fourier transform pair is

$$\Phi_n(\underline{v}) = \int_{\mathbb{R}^n} e^{-j2\pi \underline{v} \cdot \underline{r}} B_n(\underline{r}) d^n r \quad (5a)$$

$$B_n(\underline{r}) = \int_{\mathbb{R}^n} e^{j2\pi \underline{v} \cdot \underline{r}} \Phi_n(\underline{v}) d^n v \quad (5b)$$

where the subscript n denotes the dimension of the space in which the transform is defined and the spatial frequency vector $\underline{v}$ which is

equal to the scattering vector $\underline{K}$ divided by 2π has been employed to eliminate a scale factor of $(2\pi)^{-n}$ otherwise required in front of one of the integrals. Since these expressions provide a compact notation to treat spaces of different dimensions that are being considered and also facilitate understanding of the general relations that exist between scattering in 1, 2, and 3 dimensions as well as in spaces of higher dimensions, the analysis that follows will employ them rather than the individual two- and three-dimensional expressions (Eq.1 and 2). The n -dimensional Fourier transforms in this context relate the power spectrum and the correlation function of a homogeneous or statistically stationary random process defined in a space of n dimensions.

When the medium is isotropic, i.e. the statistical properties are independent of direction, the correlation function is rotationally invariant and can be written as

$$B_n(r) = B(r) \tag{6}$$

This dependence of the correlation function on only a single variable allows reduction of the n -dimensional Fourier transform relation to a one-dimensional expression by changing variables to a general spherical coordinate system, separating the integration into a product of an integral over an $(n-1)$ -dimensional surface and an integral over the distance to the surface, and expressing the

surface integral in terms of a Bessel function and the surface of a unit sphere in $n-2$ dimensions. The result is (6)

$$\Phi_n(v) = 2\pi(v)^{-\left(\frac{n-2}{2}\right)} \int_0^\infty r^{\frac{n}{2}} B(r) J_{\frac{n-2}{2}}(2\pi v r) dr \quad (7)$$

where

$$v = (v_1^2 + v_2^2 + \dots + v_n^2)^{\frac{1}{2}}$$

$$r = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

n = dimension of space

and $J_{\frac{n-2}{2}}(\beta)$ is a Bessel function defined by the integral relation

$$J_{\frac{n-2}{2}}(\beta) = \left(\frac{\beta}{2\pi}\right)^{\frac{n-2}{2}} \Omega_{n-1} \frac{1}{2\pi} \int_{-1}^1 e^{j\beta\xi} (1-\xi^2)^{\frac{n-3}{2}} d\xi$$

The quantity Ω_k in this Bessel function expression is the surface of a unit sphere in a space of k dimensions and can be expressed recursively in terms of the surface area in spaces of lower dimension as

$$\Omega_k = \Omega_{k-1} 2 \int_0^{\pi/2} (\sin \Theta)^{k-2} d\Theta \quad (8a)$$

or in terms of the gamma function

as

$$\Omega_k = \frac{2(\pi)^{k/2}}{\Gamma(\frac{k}{2})} \quad (8b)$$

where

$$\Gamma\left(\frac{k}{2}\right) = \begin{cases} 1 \cdot 2 \cdot 3 \cdot \dots \cdot \left(\frac{k}{2}-1\right) & ; k \text{ even} \\ 1 \cdot 3 \cdot 5 \cdot \dots \cdot (k-2) \frac{\sqrt{\pi}}{2^{\frac{k-1}{2}}} & ; k \text{ odd} \end{cases}$$

Since $\sin \Theta$ in the integrand of the integral expression (Eq.8a) for the surface is the distance from the axis of the coordinate being eliminated to the intersection of the unit sphere with a $(k-1)$ -dimensional planar surface normal to that axis, the integral relation for the surface can also be written

$$\Omega_k = \Omega_{k-1} 2 \int_0^1 [1 - (\hat{v})^2]^{\frac{k-3}{2}} d\hat{v} \quad (8c)$$

where $\hat{v} = \cos \Theta$

Specific expressions (Table I) for spaces of 1, 2, and 3 dimensions are readily obtained from the general formula (Eq.7) relating the power spectrum and the rotationally symmetric correlation function. The expressions for two-dimensional space are an immediate consequence of substituting $n=2$ and are known as a Fourier-Bessel transform pair* while the relations for the one- and three-dimensional space follow, respectively, from the identities (7)

$$J_{-\frac{1}{2}}(\beta) = \left(\frac{2}{\pi\beta}\right)^{\frac{1}{2}} \cos\beta \quad (9)$$

and

$$J_{\frac{1}{2}}(\beta) = \left(\frac{2}{\pi\beta}\right)^{\frac{1}{2}} \sin\beta \quad (10)$$

In one-dimensional space, the unit sphere degenerates to a line of length 2, while in two-dimensional space, the unit sphere is a circle with perimeter 2π , and in three-dimensional space the unit sphere is a sphere (in the usual solid geometry meaning of the word) with area 4π . The results for two and three dimensions have been

* see also Appendix B

Table I. One-dimensional transform pairs derived from Fourier transformation of rotationally invariant functions in spaces of one, two, and three dimensions

Dimension	Spectral Power Density	Correlation Function
n	$\Phi_n(\nu)$	$B(r)$
1	$\Phi_1(\nu) = 2 \int_0^\infty B(r) \cos(2\pi \nu r) dr$	$B(r) = 2 \int_0^\infty \Phi_1(\nu) \cos(2\pi \nu r) d\nu$
2	$\Phi_2(\nu) = 2\pi \int_0^\infty B(r) r J_0(2\pi \nu r) dr$	$B(r) = 2\pi \int_0^\infty \Phi_2(\nu) \nu J_0(2\pi \nu r) d\nu$
3	$\Phi_3(\nu) = \frac{2}{\nu^2} \int_0^\infty B(r) r \sin(2\pi \nu r) dr$	$B(r) = \frac{2}{\nu^2} \int_0^\infty \Phi_3(\nu) \nu \sin(2\pi \nu r) d\nu$

obtained previously by others (2,3,5) who dealt with spaces of fixed dimension and did not consider the general n-dimensional situation.

There are additional relations that can be obtained between the power spectra of rotationally invariant functions by writing a power spectrum defined in (n-k)-dimensional space in terms of an integral over a power spectrum defined in an n-dimensional space. This integration can be interpreted geometrically as a projection operation. The results are developed by starting with

$$\Phi_{n-k}(\mathcal{V}) = \int_{\mathbb{R}^k} \Phi_n(\sqrt{\mathcal{V}^2 + (\mathcal{V}')^2}) d^k \mathcal{V}' \quad (11)$$

where

$$\mathcal{V} = (\mathcal{V}_1^2 + \mathcal{V}_2^2 + \dots + \mathcal{V}_{n-k}^2)^{1/2}$$

and

$$\mathcal{V}' = (\mathcal{V}_{n-k+1}^2 + \mathcal{V}_{n-k+2}^2 + \dots + \mathcal{V}_n^2)^{1/2}$$

This integration can be rewritten using the same steps as those employed to simplify the Fourier transform relation (Eq. 5a) in n dimensions. The outcome in this case is

$$\Phi_{n-k}(v) = \int_0^{\infty} (v')^{k-1} \Phi_n(\sqrt{v^2 + (v')^2}) \Omega_k dv' \quad (12)$$

Then, the change of variables

$$\hat{v} = \sqrt{v^2 + (v')^2}$$

allows the integral to be expressed in terms of an integration over $\hat{v}$ as

$$\Phi_{n-k}(v) = \Omega_k \int_v^{\infty} \Phi_n(\hat{v}) [(\hat{v})^2 - v^2]^{\frac{k-2}{2}} \hat{v} d\hat{v} \quad (13)$$

Explicit formulas (Table II) can be found from this expression to relate the power spectra in 1, 2, and 3 dimensions by noting (from Equations 8a, b, or c) that

$$\Omega_1 = 2$$

and

$$\Omega_2 = 2\pi$$

Table II. Projection relations between power spectra in spaces of one, two, and three dimensions

$$\Phi_2(\nu) = 2 \int_{\nu}^{\infty} \Phi_3(\hat{\nu}) \frac{\hat{\nu}}{[(\hat{\nu})^2 - (\nu)^2]^{\frac{1}{2}}} d\hat{\nu}$$

$$\Phi_1(\nu) = 2 \int_{\nu}^{\infty} \Phi_2(\hat{\nu}) \frac{\hat{\nu}}{[(\hat{\nu})^2 - (\nu)^2]^{\frac{1}{2}}} d\hat{\nu}$$

$$\Phi_1(\nu) = 2 \pi \int_{\nu}^{\infty} \Phi_3(\hat{\nu}) \hat{\nu} d\hat{\nu}$$

(Alternative derivation, see Appendix C).

Differentiation of the relation between the power spectrum in three dimensions and in one dimension yields the expression

$$\Phi_3(v) = - \frac{1}{2\pi v} \frac{d}{dv} \Phi_1(v) \quad (14)$$

which has also been pointed out by others (3,5) as a consequence of isotropy. However, a similar relation between the power spectrum in two-dimensional space and in one-dimensional space does not generally exist because the factor $[(\hat{v})^2 - v^2]^{-1/2} \hat{v}$ becomes infinite when $\hat{v} = v$.

The foregoing relationships of the three-dimensional and two-dimensional Fourier transforms to one-dimensional expressions indicate that a volume scattering power spectrum can be calculated in two steps from the power spectrum of scattering by a plane when the medium is isotropic. The steps are:

- (1) Convert the wavenumber power spectrum of the scattering by a plane into a one-dimensional wavenumber power spectrum.
- (2) Differentiate the one-dimensional power spectrum and divide by a ramp function corresponding to spatial frequency vector magnitude times 2π .

Step (1) may be carried out by using the Fourier projection relations, i.e. summing, for example, the vertical coordinates for each horizontal coordinate in the two-dimensional power spectrum for spatial frequency vector magnitudes from zero to the maximum value. Alternatively, the one-dimensional power spectrum can be found by first Fourier-Bessel transforming the power spectrum in two dimensions considered as a function of the magnitude of spatial frequency vector to obtain the corresponding radially symmetric correlation function and then Fourier transforming the correlation function.

The distribution of spectral power in wave space can be described by various moments. The moments of power scattered in an n -dimensional statistically isotropic medium are easily reduced in a generalized spherical coordinate system to a single integration over wavevector magnitude as a consequence of the rotational symmetry that exists. The reduction introduces a scale factor that is equal to the surface of a sphere in n -dimensional space and, then, proportional to the spatial frequency vector magnitude raised to the $(n-1)$ -st power. The product of the scale factor and the wavenumber power spectrum gives the spectral power in an $(n-1)$ -dimensional shell of radius equal to the spatial frequency vector magnitude. Integration of this product over all values of spatial frequency vector magnitude yields total power.

In view of the foregoing, an appropriate analytic expression for the q -th moment of spectral power in an n -dimensional space containing a rotationally symmetric wavenumber power spectrum is

$$m_n(q) = \frac{\int_0^{\infty} v^{q+n-1} \Phi_n(v) dv}{\int_0^{\infty} v^{n-1} \Phi_n(v) dv} \quad (15)$$

In this expression, the denominator is a normalizing factor which permits direct comparison of moments computed from different wave-number power spectra and is proportional to total power. The constant of proportionality is cancelled by an identical constant in the numerator and so is not included.

The moments in an $(n-k)$ -dimensional space derived from a rotationally symmetric power spectrum in an n -dimensional space by integration over k of the Cartesian variables can be related to the moments calculated for the n -dimensional space. The relation follows in two steps from consideration of an integral having the general form of the numerator and denominator of the moment formula. First, we recognize that the integral can be expressed in terms of an integration in a space of $n+l$ dimensions and the surface of a unit sphere in $n+l$ dimensions as

$$\int_0^\infty \Phi_n(v) v^{n+l-1} dv = \frac{1}{\Omega_{n+l}} \int_{\mathbb{R}^{n+l}} \Phi_n(v) d^{n+l}v \quad (16)$$

Second, we split the integral in $(n+l)$ -space into the product of two integrals, one spanning k dimensions and the other spanning $n-k+l$ dimensions, and introduce spherical coordinates in the subspaces. The result is

$$\int_{\mathbb{R}^{n+l}} \Phi_n(v) d^{n+l}v = \int_0^\infty \left[\int_0^\infty \Phi_n(\sqrt{(v')^2 + v^2}) (v')^{k-1} \Omega_k dv' \right] v^{n-k+l-1} \Omega_{n-k+l} dv \quad (17)$$

However, a form (Eq. 12) of the projection relation permits the bracketed term to be rewritten as

$$\int_0^\infty \Phi_n(\sqrt{(v')^2 + v^2}) (v')^{k-1} \Omega_k dv' = \Phi_{n-k}(v) \quad (18)$$

and we have the basic formula

$$\int_0^{\infty} \Phi_{n-k}(v) v^{n-k+l-1} dv = \frac{\Omega_{n+l}}{\Omega_{n-k+l}} \int_0^{\infty} \Phi_n(v) v^{n+l-1} dv \quad (19)$$

Substitution of this result with $l=q$ and $l=0$ in the expression (Eq.15) for moments and employing the gamma function representation (Eq. 8b) for the surfaces involved leads to

$$m_{n-k}(q) = \frac{\Gamma\left(\frac{n+q-k}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+q}{2}\right) \Gamma\left(\frac{n-k}{2}\right)} m_n(q) \quad (20)$$

Explicit formulas (Table III) relating the moments of spectral power in spaces of one, two, and three dimensions follow from the use of this last expression (Eq.20) with $n=3$ and $k=1,2$ and with $n=2$ and $k=1$. The formulas permit the determination of moments in three-dimensional space from those obtained in two- and one-dimensional space as well as the determination of the moments in two-dimensional space in terms of those computed for a one-dimensional space. (An alternate derivation of Eq. 20 is shown in App. D).

Expressions for the central moments are readily obtained in terms of the basic moment relation (Eq.15) by starting with the following definition of the q -th central moment of spectral power in n -dimensional space.

Table III. Relations between moments of spectral power in one, two, and three dimensions.

DIMENSIONALITY OF SPACE

	1	2,1	3,2,1
MOMENT NUMBER			
1	$m_1(1)$	$m_2(1) = \frac{\pi}{2} m_1(1)$	$m_3(1) = \frac{4}{\pi} m_2(1) = 2m_1(1)$
2	$m_1(2)$	$m_2(2) = 2m_1(2)$	$m_3(2) = \frac{3}{2} m_2(2) = 3m_1(2)$
3	$m_1(3)$	$m_2(3) = \frac{3\pi}{4} m_1(3)$	$m_3(3) = \frac{16}{3\pi} m_2(3) = 4m_1(3)$

$$S_n(q) = \frac{\int_0^{\infty} [v - m_n(1)]^q v^{n-1} \Phi_n(v) dv}{\int_0^{\infty} v^{n-1} \Phi_n(v) dv} \quad (21)$$

Evaluation of this relation as a function of n for $q = 2$ yields the second central moment or variance of the spectral power in n -dimensional space. The result is

$$S_n(2) = m_n(2) - [m_n(1)]^2 \quad (22)$$

It is well known that the square root of this quantity or standard deviation describes the spread of the distribution about the mean in units directly comparable to those of the mean.

A similar evaluation of the relation as a function of n for $q=3$ gives the third central moment of the spectral power in n -dimensional space. The result in this case is

$$S_n(3) = m_n(3) - 3 m_n(2) m_n(1) + 2 [m_n(1)]^3 \quad (23)$$

The third central moment divided by the cube of the standard deviation is known as the skewness and gives a normalized measure of the asymmetry of the distribution about the mean.

The relations (Table III) between moments of spectral power in one, two, and three dimensions can be employed in the central moment expressions (Eq.22 and Eq.23) to produce explicit formulas (Table IV) for the central moments in three-dimensional space in terms of

Table IV. Relations between central moments of spectral power and the moments around the origin in one, two, and three dimensions.

DIMENSIONALITY OF SPACE

1

2, 1

3, 2, 1

Moment Number

$$2 \quad \zeta_1(2) = m_1(2) - \{m_1(1)\}^2$$

$$\zeta_2(2) = m_2(2) - \{m_2(1)\}^2$$

$$\zeta_3(2) = m_3(2) - \{m_3(1)\}^2$$

$$= 2m_1(2) - \frac{\pi^2}{4} \{m_1(1)\}^2$$

$$= \frac{3}{2} m_2(2) - \frac{16}{\pi^2} \{m_2(1)\}^2$$

$$= 3m_1(2) - 4\{m_1(1)\}^2$$

$$3 \quad \zeta_1(3) = m_1(3) - 3m_1(2)m_1(1)$$

$$\zeta_2(3) = m_2(3) - 3m_2(2)m_2(1)$$

$$\zeta_3(3) = m_3(3) - 3m_3(2)m_3(1)$$

$$+ 2\{m_1(1)\}^3$$

$$+ 2\{m_2(1)\}^3$$

$$+ 2\{m_3(1)\}^3$$

$$= \frac{3\pi}{4} m_1(3) - 3\pi m_1(2)m_1(1)$$

$$= \frac{16}{3\pi} m_2(3) - \frac{18}{\pi} m_2(2)m_2(1)$$

$$+ \frac{\pi^3}{4} \{m_1(1)\}^3$$

$$+ \frac{128}{\pi^3} \{m_2(1)\}^3$$

$$= 4m_1(3) - 18m_1(2)m_1(1)$$

$$+ \{16 m_1(1)\}^3$$

the moments in two or one dimensions as well as for the central moments in two-dimensional space in terms of the moments in one dimension. These relations between moments may be used alone to describe the power spectrum in three dimensions when the differentiation process involved in the point-by-point determination of values can not be carried out precisely due to noise in the data.

A good estimate of the power spectrum of scattering by a plane can be obtained by Fourier transformation of a single sample function when the aperture is sufficiently large in comparison to the size and distribution of objects that virtually the same pattern is obtained from other sample functions. However, for scales and apertures of interest in ultrasound characterization of tissue, the aperture may not contain a sufficient number of scatterers to produce an estimate with an acceptably small variance when the Fourier transform of a single sample function is used. This limitation can be overcome by employing an average of the spectra, also known as periodograms, obtained from a number of sample functions. An analytic expression characterizing this averaging process is

$$\Phi_P(\underline{v}) = \frac{1}{N} \sum_{i=1}^N |\Psi_i(\underline{v})|^2 \quad (24)$$

where

$$\Psi_i(\underline{v}) = \int_A \Psi_i(\underline{r}) e^{j2\pi \underline{v} \cdot \underline{r}} d^2 r$$

and $\Psi_i(\underline{r})$ is an appropriately normalized function describing wave transmission by the plane.

An estimate of the distribution of scattered power as a function of spatial frequency magnitude in wave space can be obtained by averaging radial lines taken at increments within the two-dimensional wavenumber power spectrum considered with 0 frequency in the center. An expression for this estimate which can be used as a starting point in the computation of volume scattering is

$$\bar{\Phi}_p(v) = \frac{1}{M} \sum_{m=1}^M \Phi_p(v) \Big|_{v=r_m} \quad (25)$$

where r_m is the m-th radius taken from the two-dimensional spectral estimate.

METHOD

A series of computer programs were employed to implement the processing required to obtain volume scattering power spectra from the wavenumber power spectra of scattering by planes and to find moments of the various spectra which were represented with integer values in all cases. First, the Fourier-Bessel transform of the distribution of power produced by scattering from a plane and considered as a function of spatial frequency magnitude was computed using an integer transform routine. This yielded a one-dimensional correlation function giving a spatial coordinate description of the scattering by a line.

The one-dimensional correlation function was then weighted by a Blackman window (8) to produce a function whose Fourier transform was monotonically decreasing. The span or nonzero interval of the window was chosen to reduce statistical uncertainty and produce the required non-increasing character of the one-dimensional power spectrum in the presence of noise and discrepancies between actual images employed and ideal planar sections of the volumes. The one-dimensional power spectrum was obtained by Fourier transform of the weighted one-dimensional correlation function using an integer fast Fourier transform routine.

As a check, the one-dimensional wavenumber power spectrum was also calculated by summing the y-axis values of the two-dimensional power spectra as a function of position along the x-axis to perform the necessary projection operation. The smoothing in this case was accomplished by a convolution of the resulting one-dimensional

wavenumber power spectrum with the Fourier transform of the same window that was employed in the first calculation.

To complete the processing, the one-dimensional power spectrum of scattering by a line was converted into a volume scattering power spectrum by a sequence of three steps. First, the one-dimensional power spectrum was differentiated using a frequency domain filter function that provided a variable span over which the differentiation was carried out and also provided the option of Blackman window weighting. The span and Blackman window weighting were chosen to eliminate the effects of discontinuities in the one-dimensional power spectrum or residual spurious fluctuations arising from noise in the data from which it was derived. Next, the result of this differentiation was divided by a ramp function corresponding to scattering vector magnitude. Then, the value of the dc point, which is indeterminate if the zero value of the derivative and the ramp at that point are divided, was obtained by an implementation of L'Hopital's rule which indicates that such an indeterminate expression may be evaluated by differentiating both the numerator and denominator and forming a new ratio.

The power spectrum of scattering by a plane was estimated as a function of spatial frequency vector magnitude from the two-dimensional spectra of a number of images (typically 12 or 24). An integer two-dimensional fast Fourier transform program was employed to calculate the spectral magnitudes of each image. The spectra were then averaged to produce a single two-dimensional spectrum. The average spectrum as a function of spatial frequency vector

magnitude was obtained from this by averaging the spectra taken along 180 radial lines spaced 1° apart.

The first moment and the second and third central moments were computed for the windowed wavenumber power spectra in two-dimensional space and also for the wavenumber power spectra derived from it for spaces of one and three dimensions. These moments were also computed from the unwindowed two-dimensional power spectra estimates to determine the effect of windowing the correlation functions used to produce the one-dimensional spectra. To check the results of the moments calculated directly from spectra in spaces of one and three dimensions, values for the first moment and the second and third central moments were computed from the moments of the windowed spectra in two-dimensions using the relations that were derived between moments in spaces of one, two, and three dimensions as a result of the isotropy assumption.

RESULTS

The processing steps were applied to spherically symmetric two-dimensional wavenumber power spectra having analytic expressions that could be used to check the results of processing by direct evaluation in closed form. The first two, one a Gaussian function and the other a function with a power-law decay associated with an exponential correlation function produced monotonically decreasing wavenumber power spectra in two-dimensional and three-dimensional space. The third, a piecewise continuous function constructed to produce a wavenumber power spectrum uniform between two spatial frequency vector magnitudes and zero elsewhere in three-dimensional space, produced a wavenumber power spectrum with a distinct maximum in two-dimensional space.

The results of applying the processing steps and the results of obtaining the volume scattering power spectra by direct calculation were within the greater of 1% or 5 counts on a scale of 0-32,000 in the case of the Gaussian function (Fig.1) and in the case of the function having the power-law decay (Fig.2), both of which were smooth and were differentiated without Blackman window weighting in the transform domain. In the case of the piecewise continuous function (Fig.3), the one-dimensional spectrum in three dimensions was also within the greater of 1% or 5 counts while the spectrum in three dimensions has somewhat rounded edges as a consequence of the 165 harmonic span of the Blackman window used in the transform domain implementation of the differentiation process to eliminate Gibbs phenomena arising from discontinuities of the

$$\phi_0(v) = A_0 e^{-1/2 \sigma^2 (2\pi v)^2}$$

$$A_0 = 32,000$$

$$\sigma = 1.1$$

$$\phi_1(v) = \frac{1-\pi}{\sigma} A_0 e^{-1/2 \sigma^2 (2\pi v)^2}$$

$$\phi_3(v) = \frac{\sigma}{1-\pi} A_0 e^{-1/2 \sigma^2 (2\pi v)^2}$$

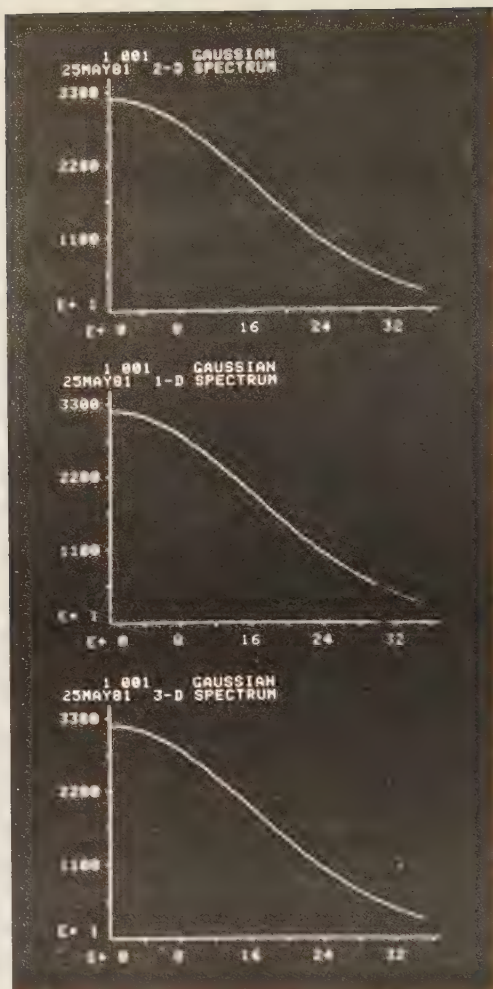


Figure 1. Gaussian Wavenumber Power Spectra. When the wavenumber power spectrum of scattering by a plane has a Gaussian form (upper left) determined by a Gaussian correlation function, analytic expressions which are also Gaussian can be derived for the spectral power distributions of line scattering (center left) and volume scattering (lower left). These expressions yield curves which are very close to those determined by processing the numerical values for the wavenumber power spectrum of scattering by a plane (upper right) to obtain the power spectrum of line scattering (center right) and volume scattering (lower right). (In the plots for one and three dimensions, the maximum amplitude has been scaled to 32,000. In all the plots, the abscissa is harmonic number or spatial frequency in cycles per unit length.)

$$\phi_2(v) = \frac{A_0}{[1+(r_0^2 v^2)]^{3/2}}$$

$$A_0 = 32,000$$

$$r_0 = .015$$

$$\phi_1(v) = \frac{(2 A_0 / r_0)}{[1+(r_0^2 v^2)]}$$

$$\phi_3(v) = \frac{(2 r_0 A_0 / \pi)}{[1+(r_0^2 v^2)]^2}$$

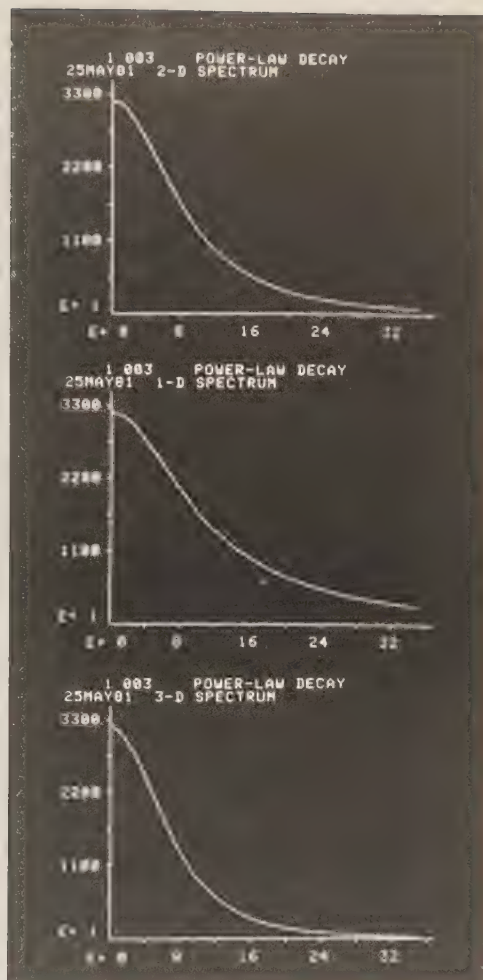


Figure 2. Power-Law Wavenumber Power Spectra. When the wavenumber power spectrum of scattering by a plane has a power-law decay (upper left) determined by an exponential correlation function, analytic expressions which exhibit different power-law decays can be derived for the wavenumber power spectrum of line scattering (center left) and volume scattering (lower left). These expressions yield curves which are very close to those determined by processing the numerical values for the wavenumber power spectrum of scattering by a plane (upper right) to obtain the wavenumber power spectrum of line scattering (center right) and volume scattering (lower right). (In the plots for one and three dimensions, the maximum amplitude has been scaled to 32,000. In all the plots, the abscissa is harmonic number or spatial frequency in cycles per unit length.)

$$S_1(v) = \int_0^1 \left[(2\pi v_2)^2 - (2\pi v)^2 \right]^{1/2} dv, \quad 0 \leq v < v_1$$

$$= \int_0^1 \left[(2\pi v_1)^2 - (2\pi v)^2 \right]^{1/2} dv, \quad v_1 \leq v < v_2$$

$$= 0, \quad v_2 \leq v < \infty$$

$$v_0 = 32,000, \quad v_1 = 0, \quad v_2 = 20$$

$$S_2(v) = \int_0^1 \left[(2\pi v_1)^2 - (2\pi v)^2 \right] dv, \quad 0 \leq v < v_1$$

$$= \int_0^1 \left[(2\pi v_1)^2 \left[1 - \left(\frac{v}{v_1} \right)^2 \right] \right] dv, \quad v_1 \leq v < v_2$$

$$= 0, \quad v_2 \leq v < \infty$$

$$S_3(v) = \int_0^1 \left[(2\pi v_1)^2 - (2\pi v)^2 \right] dv, \quad 0 \leq v < v_1$$

$$= \int_0^1 \left[(2\pi v_1)^2 \left[1 - \left(\frac{v}{v_1} \right)^2 \right] \right] dv, \quad v_1 \leq v < v_2$$

$$= 0, \quad v_2 \leq v < \infty$$

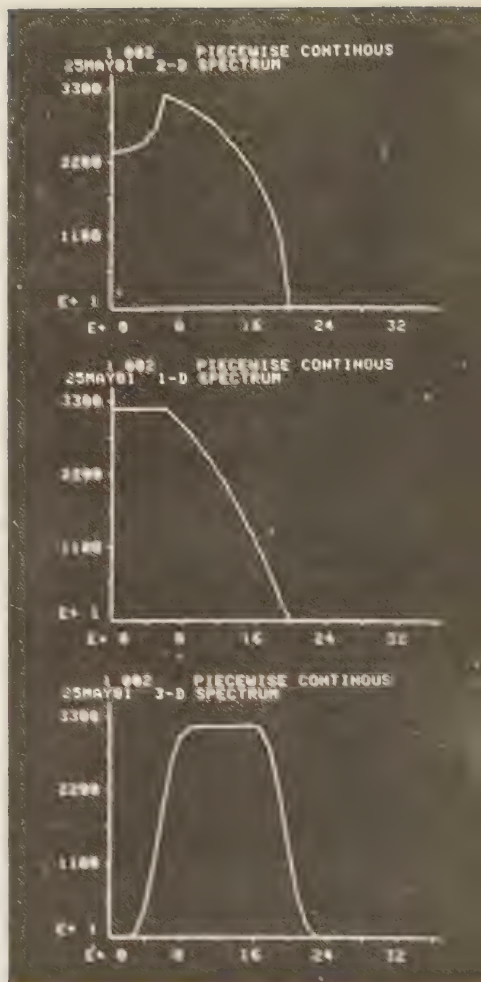


Figure 3. Piecewise Continuous Wavenumber Power Spectra. When the wavenumber power spectrum of scattering by a plane has the piecewise continuous representation (upper left) corresponding to a correlation function which is the difference between two Bessel functions, other piecewise continuous expressions are obtained for the wavenumber power spectrum line scattering (center left) and volume scattering (lower left). These expressions yield curves which are close to those determined by processing the numerical values for the wavenumber power spectrum of scattering by a plane (upper right) to obtain the wavenumber power spectrum of line scattering (center right) and volume scattering (lower right). (In the plots for one and three dimensions, the maximum amplitude has been scaled to 32,000. In all the plots, the abscissa is harmonic number of spatial frequency in cycles per unit length.)

one-dimensional spectrum.

Although our moment expressions diverge for the power spectra having a power-law decay associated with an exponential correlation function, the numerical determination of the various spectral statistics of the Gaussian and piecewise function in one-, two-, and three-dimensional space were close to the analytic values in all cases (Table VII).

The power spectrum of scattering in two dimensions was calculated for two different collections of closely packed spheres. The spectrum in two dimensions considered as a function of spatial frequency vector magnitude was then used to determine the line and volume scattering power spectra for each of the collections which were comprised of spheres with a different mean radius. The results for the spheres having the larger average radius (Fig.4) have a form qualitatively similar to the results for the spheres having the smaller average radius (Fig.5) but there are quantitative differences arising from the differing spreads of sizes around the means.

The radial distribution of power scattered by planes was also calculated for tracings of pig liver lobule boundaries and for pig liver images which were processed to enhance the lobule patterns. These radial distributions were utilized to determine the line and volume scattering power spectra for each case. Results obtained from the tracings (Fig.6) contain pronounced peaks in the spectra of scattering by both planes and volumes due to the regularity of the lobule pattern while the data produced by the enhanced versions of

Table V. Moments of Gaussian wavenumber power spectra specified in Figure 1.

Moment	DIMENSIONALITY OF SPACE (n)		
	1	2	3
$m_n(1)$	$\frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sigma}$	$\frac{\sqrt{\pi}}{\sqrt{2}} \frac{1}{\sigma}$	$2 \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sigma}$
$\zeta_n(2)$	$\frac{\pi - 2}{\pi \sigma^2}$	$\frac{4 - \pi}{2 \sigma^2}$	$\frac{3\pi - 8}{\pi \sigma^2}$
$\zeta_n(3)$	$\frac{4\sqrt{2} - \sqrt{2}\pi}{\pi \sqrt{\pi} \sigma^3}$	$\frac{\sqrt{2}\pi^2 - 3\sqrt{2}\pi}{2\sqrt{\pi} \sigma^3}$	$\frac{32\sqrt{2} - 10\sqrt{2}\pi}{\sigma^3 \pi \sqrt{\pi}}$

Table VI. Moments of piecewise smooth wavenumber power spectra specified in Figure 3.

	DIMENSIONALITY OF SPACE (n)		
	1	2	3
Moment			
$m_n(1)$	$\frac{3}{8} \frac{v_2^4 - v_1^4}{v_2^3 - v_1^3}$	$\frac{3\pi}{16} \frac{v_2^4 - v_1^4}{v_2^3 - v_1^3}$	$\frac{3}{4} \frac{v_2^4 - v_1^4}{v_2^3 - v_1^3}$
$\zeta_n(2)$	$\frac{1}{5} \frac{v_2^5 - v_1^5}{v_2^3 - v_1^3} - \frac{9}{64} \frac{(v_2^4 - v_1^4)^2}{(v_2^3 - v_1^3)^2}$	$\frac{2}{5} \frac{v_2^5 - v_1^5}{v_2^3 - v_1^3} - \frac{9\pi^2}{160} \frac{(v_2^4 - v_1^4)^2}{(v_2^3 - v_1^3)^2}$	$\frac{3}{5} \frac{v_2^5 - v_1^5}{v_2^3 - v_1^3} - \frac{9}{16} \frac{(v_2^4 - v_1^4)^2}{(v_2^3 - v_1^3)^2}$
$\zeta_n(3)$	$\frac{1}{8} \frac{v_2^6 - v_1^6}{v_2^3 - v_1^3} - \frac{9}{32} \frac{(v_2^4 - v_1^4)(v_2^5 - v_1^5)}{(v_2^3 - v_1^3)^2} + \frac{9}{128} \frac{(v_2^4 - v_1^4)^2(v_2^5 - v_1^5)}{(v_2^3 - v_1^3)^3}$	$\frac{3\pi}{32} \frac{v_2^6 - v_1^6}{v_2^3 - v_1^3} - \frac{9\pi}{40} \frac{(v_2^4 - v_1^4)(v_2^5 - v_1^5)}{(v_2^3 - v_1^3)^2} + \frac{27\pi^3}{2048} \frac{(v_2^4 - v_1^4)^3}{(v_2^3 - v_1^3)^3}$	$\frac{1}{2} \frac{v_2^6 - v_1^6}{v_2^3 - v_1^3} - \frac{27}{20} \frac{(v_2^4 - v_1^4)(v_2^5 - v_1^5)}{(v_2^3 - v_1^3)^2} + \frac{27}{32} \frac{(v_2^4 - v_1^4)^3}{(v_2^3 - v_1^3)^3}$

Table VII. Spectral power density statistics determined using Equations 15 and 20 with values of Gaussian and piecewise continuous data plotted in Figures 1 and 3 respectively and spectral power density statistics predicted using closed-form formulas given in Tables V and VI respectively. ($\sigma = .01$ and $\nu_1 = 6$, $\nu_2 = 20$. Mean and standard deviation expressed in terms of harmonic number. Predictions in parentheses.)

Sample	Gaussian			Piecewise continuous		
Spatial Dimension	1	2	3	1	2	3
Mean	12.4 (12.7)	19.9 (19.9)	25.4 (25.4)	7.4 (7.6)	11.9 (12.0)	15.3 (15.3)
Standard Deviation	9.68 (9.59)	10.4 (10.4)	10.7 (10.7)	4.98 (4.86)	4.40 (4.45)	3.70 (3.45)
Skewness	0.98 (1.00)	0.63 (0.63)	0.47 (0.49)	0.34 (0.35)	-0.30 (-0.30)	-0.54 (-0.68)

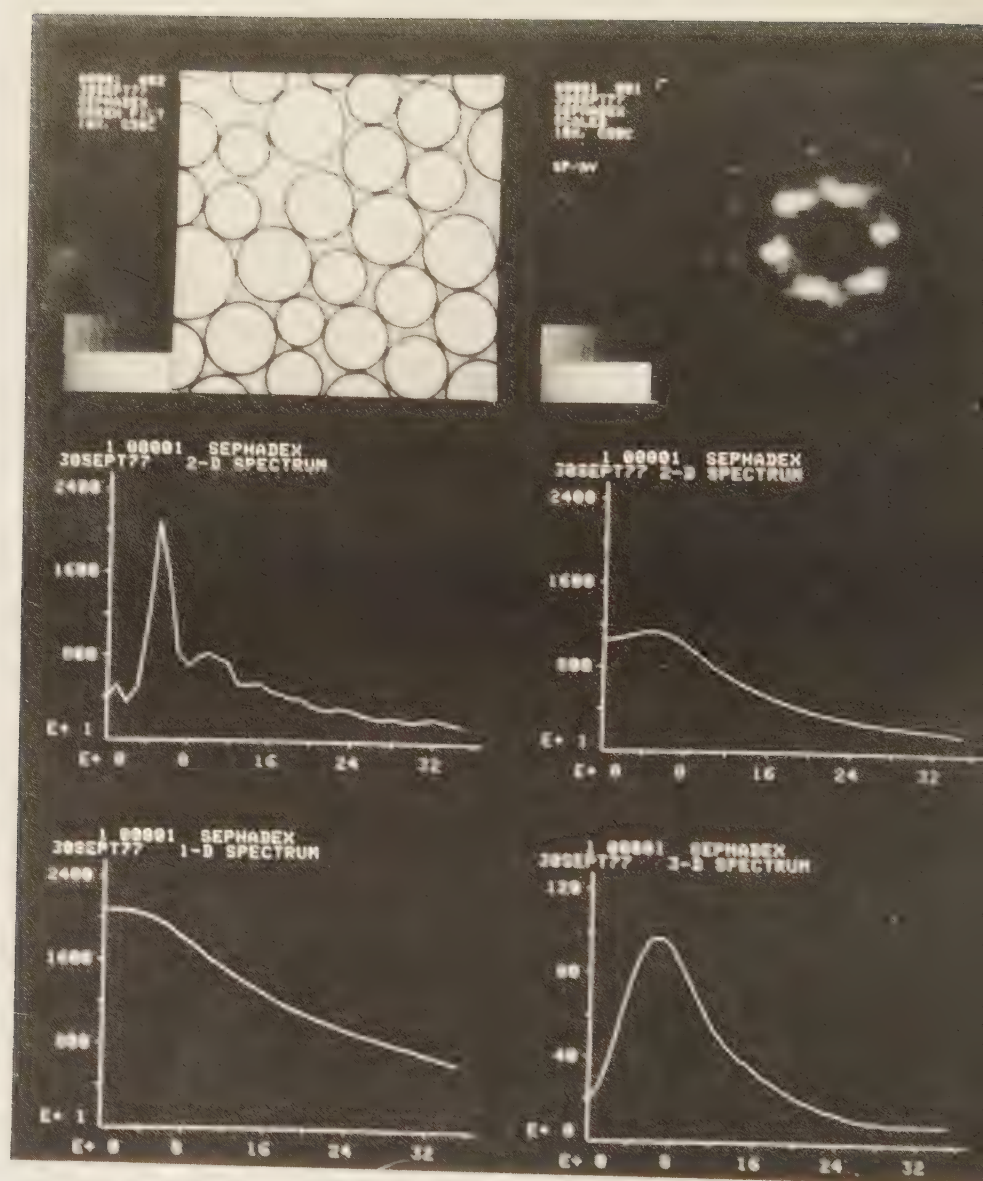


Figure 4.

Figure 4. Large Closely Packed Sphere Data. A typical two-dimensional image (upper left) shows the outlines of the spheres as dark rings against a light background. An average of 12 image periodograms with dc value suppressed yields a two-dimensional spectral power estimate (upper right) containing a circular ring of energy that indicates a more frequent occurrence of one size sphere. The radial distribution of spectral power (center left) in two dimensions was found by averaging data at corresponding points along lines radiating at 1° increments from the center. Convolution with a Blackman window 81 harmonics wide in the domain of the correlation function produces a smoothed radial distribution (center right) to characterize the scattering by a plane from which the monotonically decreasing spectrum (lower left) in one dimension was computed to describe scattering by a line. The power spectrum of volume scattering (lower right) which was derived from the line scattering spectrum by differentiation with a Blackman window weight spanning 97 in the transform domain and division by a ramp shows a peak at a spatial frequency inversely proportional to the average diameter or spacing in the closely packed collection. (The field size of the image is 1.13×1.13 mm and the two-dimensional spectrum is displayed on a linear amplitude scale with the horizontal distance from the center point to the edge of the field equal to 16 harmonics or 14.2 cycles/mm. The Cartesian plots show spectral power density on a linear amplitude scale as a function of spatial frequency harmonic with one major division or four harmonics equal to 3.54 cycles/mm. The amplitude of the spectrum in two dimensions is the result of a FFT scale factor chosen to produce maximum dynamic range when the dc value is suppressed. The amplitude of the spectrum in one dimension has been scaled to have the same dynamic range as the spectrum in two dimensions. The amplitude of the spectrum in three dimensions is a direct consequence of using the displayed values of the spectrum in one dimension.)

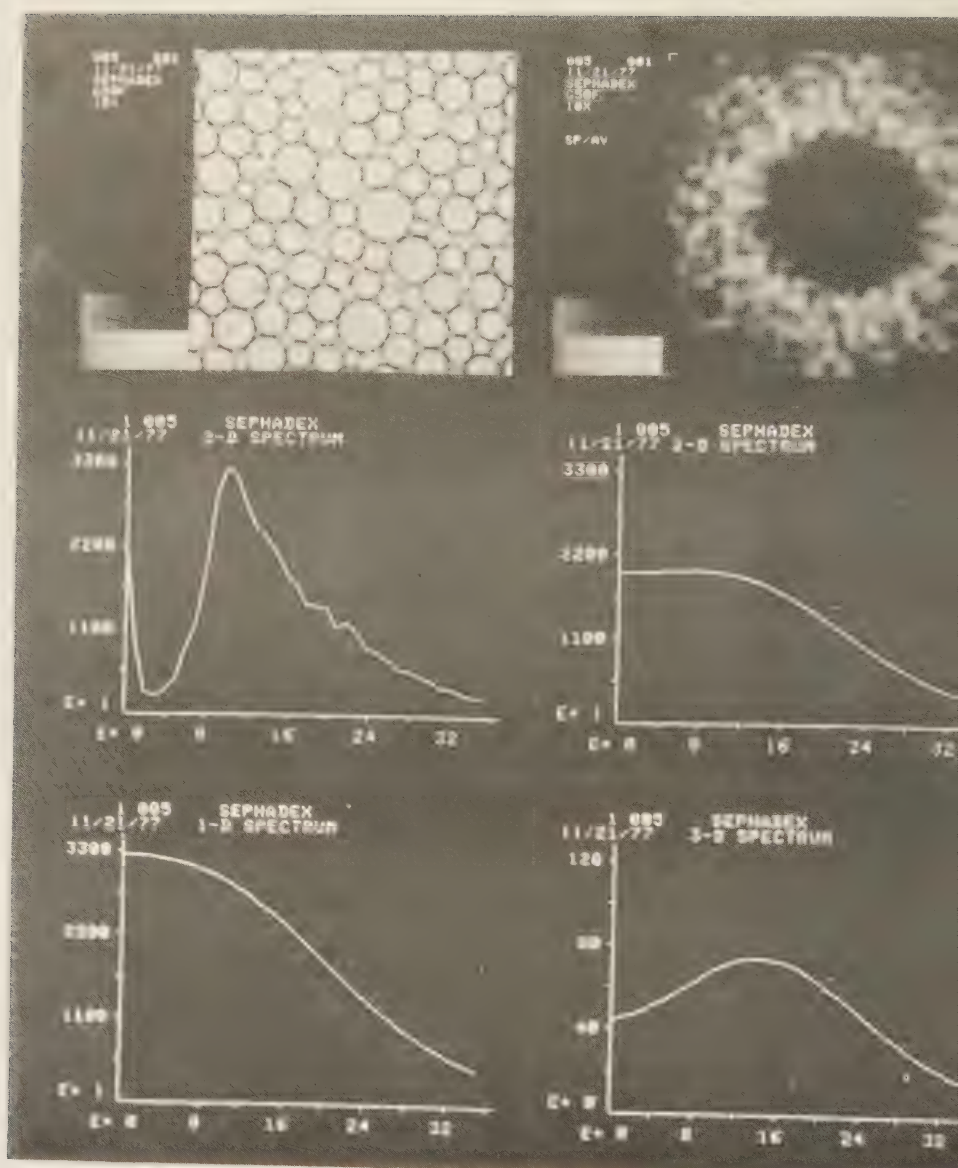


Figure 8.

Figure 5. Small Closely Packed Sphere Data. A typical two-dimensional image (upper left) shows the outlines of the spheres as dark rings against a light background. An average of 12 image periodograms with dc values suppressed yields a two-dimensional spectral power estimate (upper right) containing a circular ring of energy that indicates a more frequent occurrence of one size sphere. The radial distribution of spectral power (center left) was found by averaging data at corresponding points along lines radiating at 1° increments from the center. Convolution with a Blackman window 33 harmonics wide in the domain of the correlation function produces a smoothed radial distribution (center right) to characterize the scattering by a plane and from which the monotonically decreasing spectrum (lower left) in one dimension was computed to describe the scattering by a line. The power spectrum of volume scattering (lower right) which was derived from the line scattering spectrum by differentiation with a Blackman window weight 39 harmonics wide in the transform domain and division by a ramp shows a peak at a spatial frequency inversely proportional to the average diameter or spacing in the closely packed collection. (The field size of the image is $1.13 \times 1.13 \text{ mm}^2$ and the two-dimensional spectrum is displayed on a linear amplitude scale with the horizontal distance from the center point to the edge of the field equal to 16 harmonics or 14.2 cycles/mm. The Cartesian plots show spectral power density on a linear amplitude scale and a function of spatial frequency harmonic with one major division or four harmonics equal to 3.54 cycles/mm. The amplitude of the spectrum in two dimensions is the result of an FFT scale factor chosen to produce maximum dynamic range when the dc value is suppressed. The amplitude of the spectrum in one dimension has been scaled to have the same dynamic range as the spectrum in two dimensions. The amplitude of the spectrum in three dimensions is a direct consequence of using the displayed value of the spectrum in one dimension.)

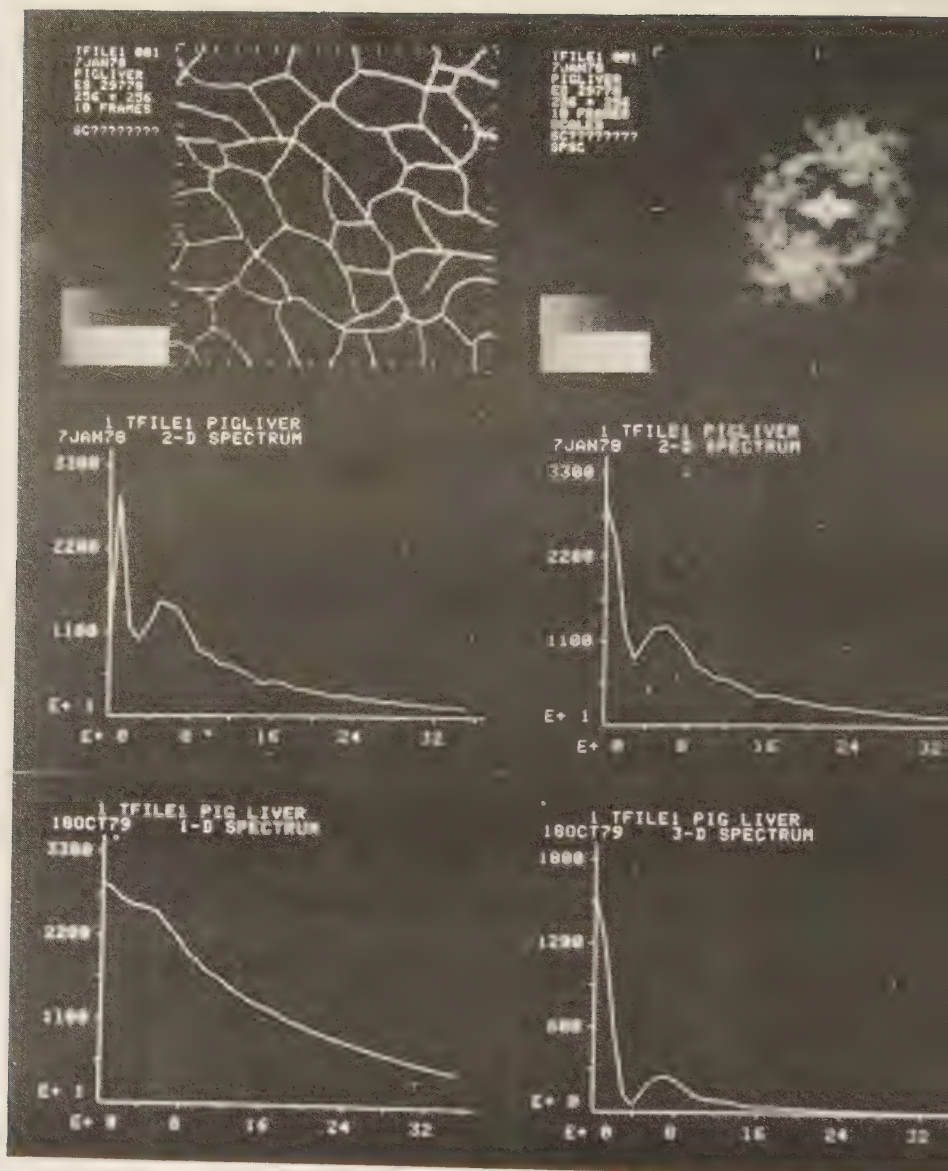
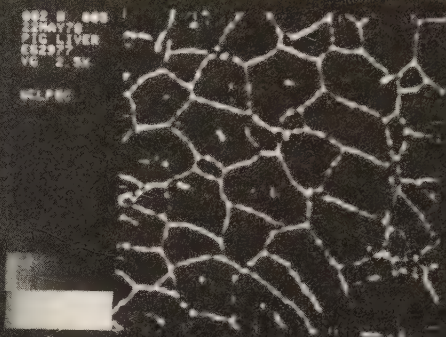


Figure 6.

Figure 6. Pig Liver Lobule Tracing Data. A typical two-dimensional image (upper left) shows the outlines of the lobules as white lines on a black background. An average of 24 periodograms with dc values suppressed yields a two-dimensional spectral power estimate (upper right) containing an ellipsoidal ring of energy that indicates a more frequent occurrence of one size lobule whose dimensions vary with direction. The radial distribution of spectral power (center left) was found by averaging data at corresponding points along lines radiating at 1° increments from the center. Convolution with a Blackman window 255 harmonics wide in the domain of the correlation function produces a smoothed radial distribution (center right) to characterize the scattering by a plane and from which the monotonically decreasing spectrum (lower left) in one dimension was computed to describe the scattering by a line. The power spectrum of volume scattering (lower right) which was derived from the line scattering spectrum by differentiation with a Blackman window weight 173 harmonics wide in the transform domain and division by a ramp shows a peak at a spatial frequency inversely proportional to the average size of the pig liver lobules. (The field size of the image is $4.4 \times 4.4 \text{ mm}^2$ and the two-dimensional spectrum is displayed on a linear amplitude scale with the horizontal distance from the center point to the edge of the field equal to 16 harmonics or 3.65 cycles/mm. The Cartesian plots show spectral power density on a linear amplitude scale as a function of spatial frequency harmonic with one major division or four harmonics equal to 0.91 cycles/mm. The amplitude of the spectrum in two dimensions is the result of an FFT scale factor chosen to produce maximum dynamic range when the dc value is suppressed. The amplitude of the spectrum in one dimension has been scaled to have the same dynamic range as the spectrum in two dimensions. The amplitude of the spectrum in three dimensions is a direct consequence of using the displayed values of the spectrum in one dimension.)

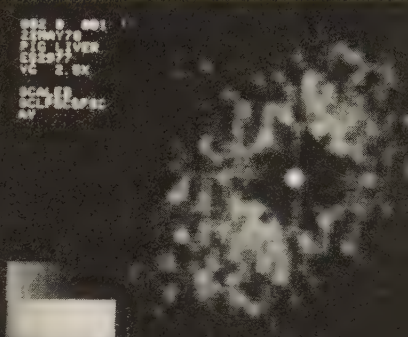
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SCALE

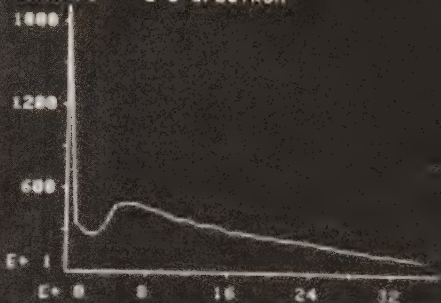


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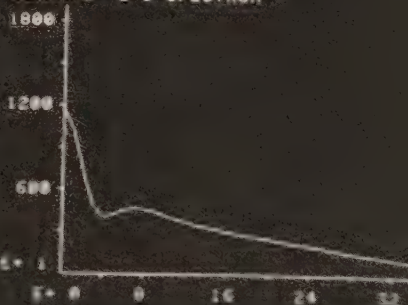
SCALE
SELF SCALING
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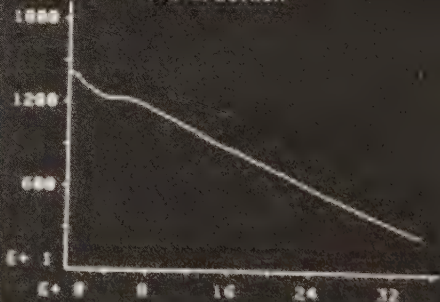
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1 002 B PIG LIVER
22MAY78 3-D SPECTRUM

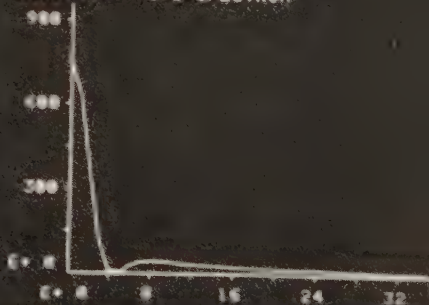


Figure 7. Enhanced Pig Liver Lobule Image Data. A typical enhanced two-dimensional image (upper left) shows the outlines of the lobules as broken light lines along with other structural features on a dark background. An average of periodograms with dc value suppressed from 24 enhanced images yields a two-dimensional spectral power estimate (upper right) containing energy from background features and an ellipsoidal ring of energy indicates a more frequent occurrence of one size lobule whose dimensions vary with direction. The radial distribution of spectral power (center left) was found by averaging data at corresponding points along lines radiating at 1° increments from the center. Convolution with a Blackman window 157 harmonics wide in the correlation function domain produces a smoothed radial distribution (center right) to characterize the scattering by a plane and from which the monotonically decreasing spectrum (lower left) in one dimension was computed to describe scattering by a line. The power spectrum of volume scattering (lower right) which was derived from the line scattering spectrum by differentiation with a Blackman window weight 149 harmonics wide in the transform domain and division by a ramp shows a peak at a spatial frequency inversely proportional to the average size of the pig liver lobules as well as the contribution of background elements that scatter. (The field size of the image is $4.4 \times 4.4 \text{ mm}^2$ and the two-dimensional spectrum is displayed on a linear amplitude scale with the horizontal distance from the center point to the edge of the field equal to 16 harmonics or 3.64 cycles/mm. The Cartesian plots show spectral power density on a linear amplitude scale as a function of spatial frequency harmonic with one major division or four harmonics equal to 0.91 cycles/mm. The amplitude of the spectrum in two dimensions is the result of an FFT scale factor chosen to produce maximum dynamic range when the dc value is suppressed. The amplitude of the spectrum in one dimension has been scaled to have the same dynamic range as the spectrum in two dimensions. The amplitude of the spectrum in three dimensions is a direct consequence of using the displayed values of the spectrum in in one dimension.

the original images (Fig.7) contain contributions from other structures which reduce the size of the peaks in the corresponding two- and three-dimensional cases.

A summary (Table VIII) of various statistics calculated from the sphere and pig liver data gives a quantitative indication of both similarities and differences among each of the model medium studies.

Table VIII.

Statistics of spectral power density derived from images of large closely packed spheres, small closely packed spheres, pig liver tracings, and enhanced pig liver data shown in Figures 4, 5, 6, and 7 respectively. Values without parentheses or brackets were determined from smoothed data using Equations 15 and 20 while values in brackets were obtained from unsmoothed data and values in parentheses were found from moments of smoothed spectra in two dimensions using formulas given in Tables III and IV. (Mean and standard deviation expressed in terms of harmonic number.)

Sample	Large Closely Packed Spheres			Small Closely Packed Spheres			Pig Liver Tracings			Enhanced Pig Liver Data		
	1	2	3	1	2	3	1	2	2	1	2	3
Spatial Dimension												
Statistic												
Mean	14.9 (13.9)	21.9 [22.9]	27.8 (27.9)	13.8 (13.1)	20.5 [21.7]	27.9 (26.0)	15.9 (16.6)	26.0 [24.7]	32.6 (33.1)	14.7 (15.3)	24.1 [23.2]	30.0 (30.7)
Standard Deviation	12.4 (11.0)	12.2 [13.2]	12.9 (12.8)	10.4 (9.38)	9.63 [10.6]	10.8 (9.27)	14.9 (15.0)	18.0 [16.9]	20.9 (20.1)	11.3 (11.3)	12.0 [11.8]	12.1 (12.1)
Skewness	1.06 (0.87)	0.32 [0.59]	0.16 [-0.03]	0.91 (0.97)	0.81 [0.51]	0.40 (1.10)	1.40 (1.39)	0.93 [0.94]	0.78 (0.68)	0.77 (0.82)	0.38 [0.34]	0.04 (0.14)

DISCUSSION

The volume scattering power spectra results obtained from analytic functions whose power spectra yield closed form expressions for volume scattering were in close agreement with the analytic values. This indicates that the processing steps were free from numerical inaccuracies within the integer range of $\pm 32,767$ for spectral and correlation function data values and integer transform routines employed in the computations. Although the special nature of the Gaussian function yields a spectrum that has the same fall-off rate for the one-, two-, and three-dimensional cases, the power-law decay associated with the exponential correlation function in one-dimension falls off less rapidly in the one-dimensional case than in the two-dimensional case which, in turn, falls off less rapidly than in the three-dimensional case. In contrast to the Gaussian and power-law decay wavenumber power spectra in two and three dimensions, the corresponding piecewise continuous wavenumber power spectra take their maxima at nonzero wavevector magnitudes. However, in all cases the statistics computed from the amplitude values of the spectra with closed form expressions compare well with the theoretical values obtained from the analytic forms.

The foregoing analytic forms of power spectra and their statistics when they exist provide references for comparison of experimentally determined values which can not be expected to be determined by convenient analytic expressions.

The computations of volume scattering power spectra for the packed spheres were made from an estimate of the radial distribu-

tion of power scattered by planes. Although only 12 frame averages were used for the two-dimensional power spectra from which the average radial distributions were estimated in the results reported here, determinations were also made of average radial distribution of scattered power from 48 frame averages of two-dimensional data. The 12 and 48 frame averages gave virtually identical values for the estimates of average radial distribution, indicating that the combination of a twelve frame average for the two-dimensional power spectra and an average of radial lines spread 1° apart was adequate for this particular model scattering medium.

Images from both collections of closely packed spheres satisfy the isotropy assumption very well since there is little directional dependence of spacing and the average of 12 periodograms determined from each collection have good circular symmetry. The amplitude peaks in the spectral ring produced by the collection of larger spheres are probably due to regularity in orientation of the triangular void formed by contact of three spheres having a similar diameter. The amplitude variations disappeared when periodograms of 48 frames were averaged.

The ring in the two-dimensional power spectrum of the spheres having the larger average size has a smaller radius than the ring in the two-dimensional power spectrum of the smaller spheres. Since a peak in spatial frequency can be interpreted as a relatively greater occurrence of the number of a specific size object per unit field, this difference in ring diameter can be viewed as a consequence of the fact that fewer of the larger spheres fit into the field of view

which is the same in both cases. The smaller diameter spectral ring produced by the larger particles implies greater small angle scattering than for the smaller size particles as expected from the well known scattering theory principle that larger or widely spaced particles scatter with a relatively higher amplitude in the forward direction.

The width of the spectral ring is less for the collection of larger size spheres than for the collection of smaller size spheres. This indicates less variability in the sizes of the larger spheres than the smaller ones. However, as the statistics show, a narrower distribution of wavenumber power spectral amplitude e does not necessarily imply that there is less spread of power in two or more dimensions because there is a weighting factor proportional to the spatial frequency vector magnitude raised to the $(n-1)$ -st power in the spectral power moment expression.

The one-dimensional correlation function of variations along a line computed for the closely packed spheres from the radial distribution of the power spectrum using a Fourier-Bessel transform is an estimate of a function whose Fourier transform, i.e. power spectrum must not only be non-negative but also monotonically decreasing. The non-negativity arises from the fact that any power spectrum by its very definition must be greater than or equal in zero. The monotonicity constraint is imposed by the requirement that a volume scattering power spectrum (which also must be non-negative since it too is a power spectrum) is proportional to the

derivative of the line scattering spectrum.

The line scattering power spectra calculated for the closely packed spheres without using a window were non-negative but each had a strong local maximum approximately at the location of the peak in the two-dimensional spectrum. The presence of this maximum caused the derivative of the line scattering spectrum to have negative values which, in turn, produce obviously spurious values in the volume scattering power spectrum. The main reason for the negative values appears to be that the two-dimensional images from which the power spectra were computed were not true cross sections of volumes because the volumes consisted of closely packed particles in a water medium and the two-dimensional images were produced by allowing a droplet of the volume to spread out into a layer about one sphere thick on a microscope slide. Therefore, the correlation functions were weighted with a Blackman window having the widest span which would yield monotonicity in the one-dimensional line scattering power spectrum. Although the resultant smoothing of the two-dimensional power spectrum almost completely removes the ring structure, the statistics are changed only slightly.

The one-dimensional spectral power density obtained for scattering along a line in the collection of spheres with a larger diameter has a larger concentration of lower spatial frequencies than the line scattering spectrum associated with the smaller spheres. The line scattering spectral power density for the larger

spheres does, however, decrease more slowly at higher wavenumbers or larger scattering angles than the corresponding spectrum of the smaller spheres. This can be viewed as another consequence of the principle that the larger spheres scatter a relatively higher amplitude in the forward direction because the slower decrease produces a smaller value for the volume scattering power spectrum which is proportional to the derivative of the line scattering spectral power density.

The two-dimensional power spectra in both studies of scattering by pig liver lobules were computed in the same way as those computed for the closely packed spheres except that periodograms of 24 frames were averaged instead of 12. When two-dimensional periodograms at 12 images of pig liver lobules were averaged, the ring structure in the two-dimensional power spectra was not as evident although the ring could still be discerned. The greater number of frames was required for estimation of the two-dimensional power spectra because the pig liver lobule liver patterns used in this study were not as regular as the closely packed spheres and more data was required to reduce statistical fluctuations that otherwise would obscure the ring.

The two-dimensional power spectra in both pig liver lobule studies have an ellipsoidal shape which indicates a larger spacing in the direction of the minor axis of the ellipse than in the direction of the major axis. Although both the tracings and the emphasized images show the same ellipsoidal shape, the ring is not as prominent in the emphasized images because they contain other

features that also contribute substantially to the two-dimensional power spectrum. These features also account for the greater spread of energy present in the power spectrum of the emphasized images.

The isotropy of the pig liver images is not as good as that of the closely packed spheres. Although the typical two-dimensional image has a suggestion of greater spacing along a direction from the upper left to the lower right than along the orthogonal direction, the presence of anisotropy is more clearly shown in the estimate of the two-dimensional power spectrum. However, the spatial frequency averaging process used to determine the two-dimensional spectral power in wave number magnitude produced a single radial distribution by weighting the peaks at different radial distances according to the number of times the peak occurred at a particular place.

The one-dimensional correlation functions describing scattering along a line were computed for the pig liver lobule patterns using a Fourier-Bessel transform in a manner similar to that already described for the collections of closely packed spheres. As in each computation for the spheres, the one-dimensional line scattering correlation function was weighted with a Blackman window having a span that would produce a monotonically decreasing one-dimensional power spectrum. However, the widths in the case of the pig liver were substantially greater and served primarily to smooth out statistical fluctuations because the solid nature of the pig liver tissue permitted slicing that yielded very good volume cross sections in comparison to those in the case of the spheres.

The wavenumber power spectra derived from the closely packed spheres and the pig liver have certain similarities. Both the sphere and pig liver spectral data in two and three dimensions have a dominant spatial frequency due to local or nearest neighbor order.

Although the spectra for the closely packed spheres were from images of black lines on a white background and the pig liver lobule spectra were derived from images composed of white lines on a black background, the general form for the results of each are the same except at zero spatial frequency. This is because one set of images can be viewed as a constant value minus the other. Therefore, the effect of shadowing or scattering can be given the same interpretation.

There are quantitative differences in the power spectra data derived from the spheres and the pig liver. Both the shapes of the power spectra and various statistics of spectral power are significantly different in one, two, and three dimensions indicating the importance of taking into account the differences in spatial dimension when making quantitative comparisons of scattering by a volume with scattering from a plane or with scattering by a line. However, there are only small differences between some large and small sphere spectral power statistics in spaces of the same dimension. These observations suggest quantitative correlation of acoustic volume scattering and scattering architecture determined from analysis of planes requires careful and appropriate processing of the power spectra of scattering by planes when the differences being studied are very small.

The one-dimensional line scattering power spectra could also have been computed directly from lines in the planar images. However, this approach does not conveniently display anisotropy that may be present unless the spectra are determined for various orientations, a process naturally accomplished when the two-dimensional spectra are employed. As demonstrated by the pig liver study, the anisotropy may be difficult to observe in the image but is readily demonstrated in the two-dimensional power spectra.

The processing steps that determine a power spectrum in a one-dimensional space from a power spectrum in a two-dimensional space can be used as a test to determine whether the spectrum in two dimensions can be associated with a true cross section of an isotropic random process defined in three dimensions. Should the steps yield a spectrum in one dimension which is not monotonically decreasing, the spectrum in two dimensions can not be from a cross section of an isotropic random process. Since the sphere images did produce non-negative spectra in one-dimensional space without substantial modification by a window function, efforts were directed at obtaining true cross sections by suspending the spheres in a liquid which would change into a solid and permit slicing. However, attempts that were made at this were unsuccessful. Nevertheless, the fact that the pig liver power spectrum in two dimensions required only modest windowing to get a monotonically decreasing spectrum in one dimension indicates that good cross sections can be obtained for some fluid-like scattering media such as tissue which can be readily fixed.

CONCLUSION

The wavenumber power spectrum of scattering by a volume of a statistically isotropic random medium can be computed from the wavenumber power spectrum of scattering by a plane within the volume using a sequence of steps that involve one-dimensional transformations. Moments and statistical descriptions of wavenumber power distribution in three dimensions also can be computed from moments of wavenumber power in two dimensions by employing relations that exist as a result of statistical isotropy. Use of the steps and the moment relations to obtain results from functions having both closed analytic expressions for volume scattering and data derived from actual images shows the differences and similarities present in one-dimensional, two-dimensional, and three-dimensional space wavenumber power spectra as well as illustrates ways in which spatial structure information is coded in wavenumber power spectra. The results indicate that structure in three dimensions can be described quantitatively from two-dimensional sections in terms of wavenumber power spectra when the basic isotropy condition is satisfied.

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APPENDIX A. Development of Scattered Wave Expressions

A form of the mathematical model commonly employed by those studying ultrasonic scattering by tissue is developed in terms of a wave equation for a lossless medium having a spatially varying sound speed, $c(\underline{r})$ (9,10). The equation is readily derived from conservation of mass and momentum relations and the thermodynamic equations of state under the assumptions of small acoustic perturbations, spatially varying compressibility or velocity, and time harmonic dependence. The result, expressed in terms of a harmonic pressure amplitude, ϕ , is

$$\nabla^2 \phi + \frac{\omega^2}{c^2(\underline{r})} \phi = 0 \quad (A1)$$

In this equation,

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \quad = \text{Laplacian operator}$$

ω = radian harmonic frequency

and

$$c(\underline{r}) = \frac{1}{(\kappa \rho)^{1/2}}$$

where ρ is the medium ambient density and κ is the medium com-

compressibility that is assumed to dominate the speed of sound variations. Dividing the wave function into a component ϕ_i representing an incident wave and a component ϕ_s representing the scattered wave and writing the sound speed as the sum of a steady value c_0 and a small spatial fluctuation c_1 produces an equation explicitly containing the scattered wave and the medium variations. The expression (Eq.A1) for the wavefunction can then be rewritten in an analytically tractable form by using a binomial expansion for the quantity $[c(\underline{r})]^{-2} = (c_0 + c_1)^{-2}$ and retaining only the first two terms. Rearranging the resulting equation yields

$$\nabla^2 \phi_s + k^2 \phi_s = 2k^2 \gamma_c (\phi_i + \phi_s) \quad (\text{A2})$$

where

$$k = \frac{\omega}{c_0} = \text{wavenumber}$$

$$\gamma_c = \frac{c_1}{c_0} = \text{normalized spatial fluctuation in wave speed}$$

The variation term on the right hand side of this equation can be interpreted as modifying the strength of a wave incident upon it. The incident wave does not appear on the left hand side because, by definition, it satisfies the homogeneous equation

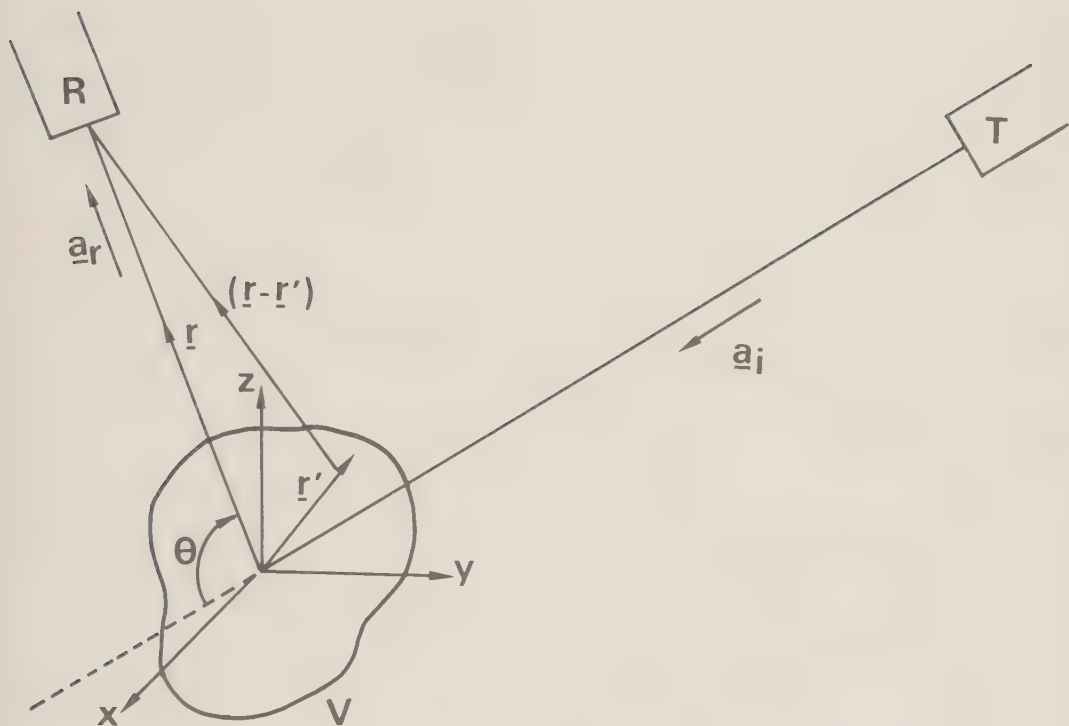


Figure A1. Scattering Geometry. A plane wave from the transmitter T travelling in the direction $\underline{a}_i$ is incident on the scattering volume (V). The scattered wave at the point receiver R is in the direction $\underline{a}_r$ and has been deflected by an angle θ from the direction of the incident wave. The receiver is at a position $\underline{r}$ and $\underline{r} - \underline{r}'$ is the vector joining R and the scattering point at $\underline{r}'$.

$$\nabla^2 \phi_i + k^2 \phi_i = 0 \quad (\text{A3})$$

An integral equation for the scattered wave at a point $\underline{r}$ can be written by treating the right hand side of the differential equation describing the scattered wave as a source term and using the Green's function for unbounded space. The result is

$$\phi_s = \frac{1}{2\pi} \int_V k^2 \chi_c(\underline{r}') \left[\phi_i(\underline{r}') + \phi_s(\underline{r}') \right] \frac{1}{|\underline{r} - \underline{r}'|} e^{jk|\underline{r} - \underline{r}'|} d^3r' \quad (\text{A4})$$

where V is the volume in which the scattering occurs. This expression for the scattered wave can be simplified at points far from the scattering volume by approximating the phase of the Green's function with the first two terms of a binomial expansion and its magnitude with $1/r$ so that

$$\frac{e^{jk|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} \longrightarrow \frac{1}{d} e^{j(kr - k\underline{a}_r \cdot \underline{r}')}$$

where

$$\underline{a}_r = \frac{\underline{r}}{r}$$

and

$$d = |\underline{r}|$$

Assuming further that the scattered wave is negligible in the scattering volume compared to the incident wave and that the incident wave is plane with the form

$$\phi_i(\underline{r}') = e^{jk \underline{a}_i \cdot \underline{r}'}$$

the expression for the scattered wave becomes

$$\phi_s(\underline{k}) = \frac{k^2 e^{jk r}}{2\pi d} \int_V e^{j\underline{k} \cdot \underline{r}'} \chi_c(\underline{r}') d^3 r' \quad (\text{A5})$$

where

$$\underline{k} = k (\underline{a}_i - \underline{a}_r)$$

and is known as the scattering vector. Thus, the scattered wave at

a receiving point can be written as the product of a frequency-dependent factor and the three-dimensional Fourier transform of the variations in wave speed. The Fourier transform or spectrum in this case is three-dimensional and defined in terms of the scattering vector $\underline{K}$ which is in the direction of the unit vector

$$\underline{a}_k = \frac{\underline{a}_i - \underline{a}_r}{|\underline{a}_i - \underline{a}_r|} \quad (A6)$$

and has magnitude

$$K = 2k \sin(\theta/2)$$

where

$$\Theta = \text{angle between } \underline{a}_i \text{ and } \underline{a}_r$$

The units of the scattering vector are those of the wavenumber and motivate our use of the terminology wavenumber power spectrum to distinguish this power spectrum from that of a time function.

The model also relates the mean square value of the scattered wave to the correlation between wave speed at points in a random medium. Writing the mean square value in terms of the scattered wave expression (Eq.A5) derived for small speed variations, single

weak scattering, plane incident wave and far-field observation yields

$$\langle |\phi_s(\underline{k})|^2 \rangle = \left\langle \frac{k^4}{4\pi^2 d^2} \iint_V e^{j\underline{k}(\underline{r}'_1 - \underline{r}'_2)} \delta_c(\underline{r}'_1) \delta_c(\underline{r}'_2) d^3 r'_1 d^3 r'_2 \right\rangle \quad (A7)$$

where $\langle \cdot \rangle$ denotes a statistical average. This equation can be rewritten by interchanging the order of integration and averaging. The result is

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4}{4\pi^2 d^2} \iint_V e^{j\underline{k}(\underline{r}'_1 - \underline{r}'_2)} B_{\delta_c}(\underline{r}'_1, \underline{r}'_2) d^3 r'_1 d^3 r'_2 \quad (A8)$$

where

$$B_{\delta_c}(\underline{r}'_1, \underline{r}'_2) = \langle \delta_c(\underline{r}'_1) \delta_c(\underline{r}'_2) \rangle$$

and is the correlation function of the wave speed variations. If the correlation function of the wave speed variations is independent of absolute position in the scattering volume and the correlation length of the scattering process is much less than the largest dimension of the scattering volume, one of the integrations can be

carried out and the mean square value of the scattered wave reduces to

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4 V}{4\pi^2 d^2} \int_V e^{j\underline{k} \cdot \underline{r}'} B_{\delta_c}(\underline{r}') d^3 r' \quad (A9)$$

where

$$B_{\delta_c}(\underline{r}') = \langle \delta_c(\underline{r}) \delta_c(\underline{r} + \underline{r}') \rangle$$

Since the average scattered intensity is proportional to the mean square value of the scattered wave and the Fourier transform of a process correlation function yields a power spectrum of that process, this Fourier transform relation shows that the mean intensity of the scattered wave is proportional to the wavenumber power spectrum of the variations in sound speed or, more briefly, the power spectrum of scattering.

Wave scattering by a plane can be described in terms of two-dimensional Fourier transform relations that are analogous to the three-dimensional Fourier transform relations. The two-dimensional relation for the scattered wave amplitude is (2)

$$\phi_s(\underline{k}) = \frac{k^2 e^{j\mathbf{k} \cdot \mathbf{r}}}{2\pi d} \int_A \delta_c(\underline{r}') e^{j\mathbf{k} \cdot \underline{r}'} d^2 r' \quad (\text{A10})$$

where

$\delta_c(\underline{r}')$ = wave speed variations within a plane

A = area producing the scattering

and the vector $\underline{r}'$ is confined to the scattering plane. The two-dimensional expression for the mean square value of a wave scattered by a plane is

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4 A}{4\pi^2 d^2} \int_A e^{j\mathbf{k} \cdot \underline{r}'} B_{\delta_c}(\underline{r}') d^2 r' \quad (\text{A11})$$

where

$$B_{\delta_c}(\underline{r}') = \langle \delta_c(\underline{r}_1) \delta_c(\underline{r}_1 + \underline{r}') \rangle$$

and is the two-dimensional correlation function of the wave speed variations in the plane. The vectors $\underline{r}_1$ and $\underline{r}'$ are two-dimensional quantities lying in the plane of variations.

APPENDIX B. Alternate Derivation of Isotropic Medium Results for Scattering.

If the 2nd-order statistical properties of a three-dimensional or volume scattering medium are independent of direction, i.e. the medium is statistically isotropic, the correlation function of wave speed variations in three dimensions can be written as a correlation along any line as

$$B_{\gamma_c}(r') = B_\ell(r') \quad (B1)$$

and the three-dimensional Fourier transform relation can be simplified to a one-dimensional transform by choosing a spherical coordinate system and integrating over the two angular variables. Then, the expression (Eq.A9) for the mean square value of the scattered wave reduces to

$$\langle |\phi_s(k)|^2 \rangle = \frac{k^4 V}{\pi d^2 k} \int_0^\infty B_\ell(r') r' \sin(kr') dr' \quad (B2)$$

This shows that the mean intensity of the scattered wave is proportional to a one-dimensional Fourier sine transform of the distance variable, r' , times the correlation function of variations in wave speed when the scattering medium is isotropic.

The sine transform may be written in terms of a derivative by defining the one-dimensional Fourier transform relation

$$\Phi_\ell(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_\ell(r') e^{jkr'} dr' \quad (B3)$$

noting that

$$\frac{d}{dk} \Phi_\ell(k) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} jr' B_\ell(r') e^{jkr'} dr'$$

and using Euler's identity. The result is

$$\langle |\Phi_s(k)|^2 \rangle = -\frac{k^4 V}{d^2} \frac{1}{2\pi k} \frac{d}{dk} \Phi_\ell(k) \quad (B4)$$

This expression, given previously in normalized form (Eq.14), shows that the mean intensity of volume scattering by an isotropic medium can be written as a frequency-dependent function times the quotient of the scattering vector magnitude and the derivative of a one-dimensional wavenumber power spectrum of scattering by a line.

If the statistical properties of a two-dimensional or planar scattering medium are independent of direction, the correlation

function in two dimensions can also be written as a function in one dimension describing the correlation of wave speed variations along any line, i.e.

$$B_{\chi_c}(\underline{r}') = B_{\chi}(r') \quad (\text{B5})$$

Then the two-dimensional Fourier transform (Eq.A11) can be reduced to a one-dimensional relation by choosing a cylindrical coordinate system and integrating over the angular variable. The result is

$$\langle |\phi_s(\underline{k})|^2 \rangle = \frac{k^4 A}{2\pi d^2} \int_0^{\infty} B_{\chi}(r') r' J_0(k r') dr' \quad (\text{B6})$$

where

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jz \cos \varphi} d\varphi$$

This indicates, as already shown in normalized form (Table I), that the mean intensity of the wave scattered by a plane is proportional to a one-dimensional Fourier-Bessel transform of the correlation function of the variations in wave speed within the plane when the planar medium is isotropic.

APPENDIX C. Alternate Derivation of Projection Relations

The relations (Table II) between power spectra defined by a rotationally invariant correlation function in spaces of one, two, and three dimensions can also be obtained by writing a power spectrum defined in (n-1)- and (n-2)-dimensional spaces in terms of an integral over a power spectrum defined in an n-dimensional space. The results are developed by starting with

$$\Phi_{n-1}(k_1, k_2, \dots, k_{n-1}) = \int_{-\infty}^{\infty} \Phi_n(k_1, k_2, \dots, k_{n-1}, k_n) dk_n \quad (C1)$$

and assuming that

$$\Phi_n(k_1, k_2, \dots, k_{n-1}, k_n) = \Phi_n(k)$$

where

$$k = (k_1^2 + k_2^2 + \dots + k_{n-1}^2 + k_n^2)^{1/2}$$

Then the change of variables

$$K_n = \left[K^2 - (K_1^2 + K_2^2 + \dots + K_{n-1}^2) \right]^{\frac{1}{2}}$$

allows the integral to be expressed in terms of an integration over K as

$$\Phi_{n-1}(K_1, K_2, \dots, K_{n-1}) = 2 \int_0^\infty \Phi_n(K) \frac{K dK}{\left[K^2 - (K_1^2 + K_2^2 + \dots + K_{n-1}^2) \right]^{\frac{1}{2}}} \quad (C2)$$

$$(K_1^2 + K_2^2 + \dots + K_{n-1}^2)^{\frac{1}{2}}$$

Since the variables $K_1, K_2, \dots, K_{n-1}$ appear in the integral only as the sum of their squares, the spectral power density in $(n-1)$ -dimensional space is also rotationally symmetric. Thus, the wavenumber power spectrum in $n-1$ dimensions can be written in terms of rotationally symmetric spectrum in n dimensions as

$$\Phi_{n-1}(\hat{K}) = 2 \int_{\hat{K}}^\infty \Phi_n(K) \frac{K dK}{(K^2 - \hat{K}^2)^{\frac{1}{2}}} \quad (C3)$$

where

$$\hat{K} = (K_1^2 + K_2^2 + \dots + K_{n-1}^2)^{\frac{1}{2}}$$

This relation between wavenumber power spectra permits the spectrum in two dimensions to be expressed in terms of the spectrum in three dimensions as

$$\bar{\Phi}_2(\hat{k}) = 2 \int_{\hat{k}}^{\infty} \Phi_3(k) \frac{k dk}{(k^2 - \hat{k}^2)^{1/2}} \quad (C4)$$

where

$$k = (k_1^2 + k_2^2 + k_3^2)^{1/2}$$

and

$$\hat{k} = (k_1^2 + k_2^2)^{1/2}$$

The analogous expression for the spectrum in one dimension in terms

of the spectrum in two dimensions is

$$\Phi_1(\hat{k}) = 2 \int_{\hat{k}}^{\infty} \Phi_2(k) \frac{k dk}{(k^2 - \hat{k}^2)^{\frac{1}{2}}} \quad (C5)$$

where, in this case,

$$k = (k_1^2 + k_2^2)^{\frac{1}{2}}$$

and

$$\hat{k} = k_1$$

An integral expression relating the wavenumber power spectrum in (n-2)-dimensional space to the spectrum in n-dimensional space follows from writing the spectrum in n-2 dimensions in terms of the spectrum in n-1 dimensions and then writing the spectrum in n-1 dimensions in terms of the spectrum in n dimensions. The initial integral equation is

$$\Phi_{n-2}(\tilde{k}) = 2 \int_{\tilde{k}}^{\infty} \frac{\hat{k}}{(\hat{k}^2 - \tilde{k}^2)^{1/2}} 2 \int_{\hat{k}}^{\infty} \Phi_n(K) \frac{K}{(K^2 - \hat{k}^2)^{1/2}} dK d\hat{k} \quad (C6)$$

where

$$K = (K_1^2 + K_2^2 + \dots + K_n^2)^{1/2}$$

$$\hat{k} = (k_1^2 + k_2^2 + \dots + k_{n-1}^2)^{1/2}$$

and

$$\tilde{k} = (k_1^2 + k_2^2 + \dots + k_{n-2}^2)^{1/2}$$

This can be simplified by interchanging the order of integration and using a series of variable changes to show that

$$\int_{\tilde{k}}^K \frac{\hat{k}}{(\hat{k}^2 - \tilde{k}^2)^{1/2} (K^2 - \hat{k}^2)^{1/2}} d\hat{k} = \frac{\pi}{2}$$

The result is

$$\Phi_{n-2}(\tilde{k}) = 2\pi \int_{\tilde{k}}^{\infty} K \Phi_n(K) dK \quad (C7)$$

APPENDIX D. Alternate Derivation of Moment Relations

The spectral power moment relations (Table III) between spaces of one, two, and three dimensions can also be obtained by starting from a substitution in the basic moment equation (Eq.15) of the expression (Eq.C3) giving the wavenumber power spectrum in (n-1)-dimensional space as an integration of the wavenumber power spectrum in n-dimensional space. The result of the substitution is

$$m_{n-1}(q) = \frac{\int_0^\infty (\hat{k})^{q+n-2} 2 \int_{\hat{k}}^\infty \Phi_n(k) \frac{k}{(k^2 - \hat{k}^2)^{1/2}} dk d\hat{k}}{\int_0^\infty (\hat{k})^{n-2} 2 \int_{\hat{k}}^\infty \Phi_n(k) \frac{k}{(k^2 - \hat{k}^2)^{1/2}} dk d\hat{k}} \quad (D1)$$

Simplification of this expression follows from interchanging the order of integration and showing with the substitution

$$\int_0^K \frac{(\hat{k})^p}{(k^2 - \hat{k}^2)^{1/2}} d\hat{k} = \begin{cases} \frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (p-1)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (p)} K^p ; p \text{ odd} \\ \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (p-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (p)} \cdot \frac{\pi}{2} K^p ; p \text{ even} \end{cases}$$

that

$$= \frac{\sqrt{\pi} (K)^p \Gamma(\frac{p}{2} + \frac{1}{2})}{2 \Gamma(\frac{p}{2} + 1)}$$

The result is that the expression (Eq.D1) for the q -th moment in $(n-1)$ -dimensional space can be written in terms of the q -th moment in n -dimensional space as

$$m_{n-1}(q) = \frac{\Gamma\left(\frac{q}{2} + \frac{n}{2} - \frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{q}{2} + \frac{n}{2}\right) \Gamma\left(\frac{n}{2} - \frac{1}{2}\right)} \cdot m_n(q) \quad (D2)$$

The moments of spectral power in an $(n-2)$ -dimensional space derived from an n -dimensional space can be obtained in terms of the moments in the n -dimensional space by recursively employing the relation (Eq.D2) obtained between the moments in $n-1$ and n dimensions. Alternatively, a procedure analogous to that employed to relate the moments in $n-1$ and n dimensions can be followed using in the basic moment definition (Eq.15), the expression (Eq.C7) for wavenumber power spectrum in $n-2$ dimensions in terms of the spectrum in n dimensions. The result in either case can be written

$$m_{n-2}(q) = \frac{(n-2)}{(q+n-2)} m_n(q) \quad (D3)$$

